I. NRGR

Here I give some detailed notes on how to calculate the 1PN potential in NRGR. Let's first work out the power counting in a theory of potentials. It is intended for readers who have little to no experience calculating Feynman diagrams, and is not familar with the whole formalism.

First we must fix the power counting. The relevant scales are

$$R_{pl}, r, M, M_{pl}. \tag{1}$$

where M is the reduced mass, R_{pl} is the radius of the object and r is the radius of the orbit. If we ignore R_{pl} for the moment. We can form two independet dimensionless parameters, which we will choose to be

$$v^2 = \frac{M}{r}\frac{1}{M_{pl}^2}, \tag{2}$$

and

$$L = \frac{M^2}{vM_{pl}^2}, \tag{3}$$

All the potential terms in the action will scale like L.

TABLE I: Scaling relations

M^2/M_{pl}^2	vL
$Md\tau$	$\frac{L}{v^2}$
h^{rad}/M_{pl}	$\frac{v^{5/2}}{\sqrt{L}}$
h^{pot}/M_{pl}	$\frac{v^2}{\sqrt{L}}$

The full theory Lagrangian is

$$S = \int -2M_{pl}^2\sqrt{g}Rd^4x \tag{4}$$

Whereas for the matter action

$$S_M = -m\int ds \tag{5}$$

where

$$ds = d\lambda\sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g^{\mu\nu}(x(\lambda))}. \tag{6}$$

This is the unique RPI invariant action. We can choose any parameter we wish.

We then expand around Minkowksi space and choose to parameterize by time.

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}} \tag{7}$$

where this is an exact relation we find

$$ds = dt\sqrt{\frac{dx^\mu}{dt}\frac{dx_\mu}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}} \tag{8}$$

Then

$$ds \approx dt(1 + \frac{1}{2}\vec{v}^2 + \frac{1}{2}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}). \tag{9}$$

so that the action is given by (note we are ignoring radiation here so the metric perturbation is pure potential)

$$S_m = -M\int dt(d\tau/dt + \frac{1}{2}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}} - \frac{1}{8}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}\frac{dx^\rho}{dt}\frac{dx^\beta}{dt}\frac{h_{\rho\beta}(x(t))}{M_{pl}} + ...) \tag{10}$$

Now we can expand each term in powers of v. Then include in the diagrams only those vertices which are relevant for the order one is interested in. This gives an effective Lagrangian which acts as an intermediae step crutch for calculating the potentials.

We are now prepared to calculate the leading order contribution to the vacuum energy, ignoring for the moment self interactions. This is done by "integrating out" the potential gravitons. Formally this means doing the path integral over potential field. But these are just words, in that we dont really do any path integration. What this really means is solve for the field and plug it back into the action to get an effective lagrangian, from which we can read off the potentials.

So we calculate all connected diagrams with no external gravitons. The interested reader who wishes to understand why this is the correct thing to do should consult Peskins textbook page 364. Note that in his notation we are interested in $Z[J]$ NOT the effective action which is the Legendre transform of Z.

I will now sketch a derivation of the fact that the potential coressponds to a sum of Feynman diagrams. For those who are just interested in plowing ahead in the calculation, this part can be skipped.

Generically,

$$Z[J] = \int D\phi e^{i\int d^4x(L(\phi)+J\phi)} \tag{11}$$

but in our case J which will the particle world lines and ϕ is the metric (in the case at hand the potential part of the metric perturbations around flat space. $Z[J]$ can be thought of as the "vacuum persistance amplitude". That is it is the probability of going from vacuum to vacuum over a large time T in the presence of a source. Quantum mechanically this means essentially

$$\langle 0 \mid 0 \rangle_J \sim \langle 0 \mid T e^{i \int L_{int}} \mid 0 \rangle. \tag{12}$$

Here T stands for time ordering and L_{int} is the interaction Lagrangian, and we are working in the so-called "interaction picture". Now to calculate we use the fact that in the EFT each term in the action scales as a definate power in v. Thus for a potential at order v^n we simply expand the exponential inlcuding all terms whose net scaling is v^n. Then we are left with an expression of the form

$$\int d^q x d^w y .... \langle 0 \mid T(h(x)^a h(y)^b ....) \mid 0 \rangle \tag{13}$$

where h is the potential (in this case) graviton and the dimensionality of the the integrations q and w depend upon whether the term in the action came from a "bulk" four dimensional interaction (these will be ONLY non-linear terms in h), or a world line term in which case the integration is over an affine parameter.

Now to calculate this time ordered product we use "Wicks Theorem", which is discussed in Peskin on page 88. This theorem tells us that any time ordered product is the sum of all possible (well ignored disconnected[1]) "Wick contractions". A contraction just means we associate pairs of fields with each other. We replace each such contraction with a propagator between the corresponding point. In a Feynman diagram this propagator is denoted by a line joining the corresponding "vertices". One can associate a space-time point with each such vertex. This is how we generate a Feynman diagrams. Expand out the exponential, and use Wick theorem. Each vertex which come with a factor, or sometimes a derivative which then acts on the propagator.

So to calculate the leading order potential, we expaned the exponent keeping only the leading order terms in v and due the Wick contractions (in this case there is only one). The

[1] Which are disconnected can be seen easily by asociating a vertex with every point and a line for any propagator connecting the points.

corresponding potential is given by

$$-iV = \int dt_1 d\tau_2 M_1 M_2 \frac{1}{4M_{pl}^2} \frac{dx_1^\mu}{dt_1} \frac{dx_1^\nu}{dt_1} \frac{dx_2^\rho}{dt_2} \frac{dx_2^\sigma}{dt_2} \langle 0 \mid T\left[h_{\mu\nu}(x_1(t_1)) h_{\rho\sigma}(x_2(t_2))\right] \mid 0\rangle. \tag{14}$$

The minus sign and factor of i is fixed most easily by just making sure we get back Newton at leading order. To calculate this we need the graviton propagator.

So we must expand

$$L = -2M_{pl}^2 \sqrt{g} R \tag{15}$$

in terms of

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}}. \tag{16}$$

We will then solve for the propagator after which we will calculate in the full theory and expand in $v_1 = m_2/(RM_{pl}^2)$. Working in the harmonic gauge the propagator is given by

$$D_{\mu\nu\alpha\beta}(q) = \frac{i}{q^2 + i\epsilon} P_{\mu\nu,\alpha,\beta} \tag{17}$$

where

$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta}) \tag{18}$$

Then we can read off the leading order potential

$$\begin{aligned} -iVT &= (-i)^2 \int dt_1 dt_2 M_1 M_2 \frac{1}{4M_{pl}^2} \langle 0 \mid T\left[h_{00}(x_1(t_1)) h_{00}(x_2(t_2))\right] \mid 0\rangle. \\ &= -\int dt_1 dt_2 M_1 M_2 \frac{1}{8M_{pl}^2} \int [d^4p] e^{ip\cdot(x_1(t_1) - x_2(t_2))} \frac{i}{p^2} \\ &= \int dt M_1 M_2 \frac{1}{8M_{pl}^2} \int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_1(t) - \vec{x}_2(t))} \frac{i}{\vec{p}^2} + ... \end{aligned} \tag{19}$$

To do this integral well insert and mass and then take it to zero when were done

$$\begin{aligned} -iV &= \lim_{m\to 0} \int dt M_1 M_2 \frac{1}{8M_{pl}^2} \int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_1(t) - \vec{x}_2(t))} \frac{i}{\vec{p}^2 + m^2} \\ &= \lim_{m\to 0} \int dt M_1 M_2 \frac{2\pi}{(2\pi)^3 8M_{pl}^2} \int p^2 dp d\cos\theta e^{-ipr\cos\theta} \frac{i}{\vec{p}^2 + m^2} \\ &= \lim_{m\to 0} \int dt M_1 M_2 \frac{2\pi}{(2\pi)^3 8M_{pl}^2} \int_0^\infty \frac{p}{-ir} dp (e^{-ipr} - e^{ipr}) \frac{i}{\vec{p}^2 + m^2} \\ &= \lim_{m\to 0} \int dt M_1 M_2 \frac{2\pi}{(2\pi)^3 8M_{pl}^2} \int_{-\infty}^\infty \frac{p}{-ir} dp (e^{-ipr}) \frac{i}{\vec{p}^2 + m^2} \\ &= -\lim_{m\to 0} \int dt M_1 M_2 \frac{2\pi}{r(2\pi)^3 8M_{pl}^2} \frac{-2\pi i(-im)}{-2im} (e^{-mr}) \\ &= \int dt \frac{M_1 M_2}{32\pi M_{pl}^2} \frac{i}{r} \end{aligned} \tag{20}$$

So that

$$V = -\frac{M_1 M_2}{32\pi M_{pl}^2}\frac{1}{r} \tag{21}$$

This potential scales like L, since $dt \propto r/v$ while $M^2/M_{pl}^2 \propto Lv$. Let us consider the subleading potentials. The best way to power count sources of potential is by using the fact that the potential graviton scales as

$$h_{pot} \propto R^{-1} v^{1/2} \tag{22}$$

To see this note that $p \propto 1/R$, $E \propto v/R$, and all units for h are in terms of R. Let's consider the leading order potential which comes from two insertions of the interaction

$$S_m = -M \int dt(1 + \frac{1}{2}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}) \tag{23}$$

which scales as (for the $0-0$ component)

$$M/M_{pl}(R/v)v^{1/2}/R = (M/M_{pl})v^{-1/2} = \sqrt{L}. \tag{24}$$

Since $M^2/M_{pl}^2 \propto Lv$. So two insertions of this operator gives Lv^0. Lets go to higher order.

$$\begin{aligned} ds^2 &= g_{00}dt^2 + 2g_{i0}dx^i dt + g_{ij}dx^i dx^j = (dt^2 + h_{00}dt^2 - dx^2 + h_{ij}dx^i dx^j + 2h_{0i}dx^i dt) \\ &= dt^2(1 + h_{00} - v^2 + v^i v^j h_{ij} + 2h_{0i}dx^i dt) \\ &= dt^2(1+h_{00})(1 - \frac{v^2}{1+h_{00}} + \frac{v^i v^j h_{ij}}{1+h_{00}} + 2\frac{v^i h_{i0}}{1+h_{00}}) \end{aligned} \tag{25}$$

So that

$$\begin{aligned} ds &= dt(1+\frac{1}{2}h_{00})(1 - \frac{1}{2}\frac{v^2}{1+h_{00}} + \frac{1}{2}\frac{v^i v^j h_{ij}}{1+h_{00}} + \frac{v^i h_{i0}}{1+h_{00}}) \\ &\approx dt(1 - \frac{1}{2}\frac{v^2}{1+h_{00}} + \frac{1}{2}h_{00} - \frac{h_{00}}{4}\frac{v^2}{1+h_{00}} + \frac{1}{2}\frac{v^i v^j h_{ij}}{1+h_{00}} + \frac{v^i h_{i0}}{1+h_{00}}) \\ &\approx dt(1 - \frac{1}{2}v^2 + \frac{1}{2}v^2 h_{00} + \frac{1}{2}h_{00} - \frac{h_{00}}{4}v^2 + \frac{1}{2}v^i v^j h_{ij} + v^i h_{i0}) \\ &\approx dt(1 + \frac{1}{2}h_{00} - \frac{1}{2}v^2 + \frac{1}{4}v^2 h_{00} + \frac{1}{2}v^i v^j h_{ij} + v^i h_{i0}) \end{aligned} \tag{26}$$

thus the kinetic energy term is

$$S_{KE} = \int dt(-M + \frac{1}{2}Mv^2) \tag{27}$$

while the linear interaction term is given by

$$S_m = -M \int (\frac{1}{2}h_{00} + \frac{1}{4}v^2 h_{00} + \frac{1}{2}v^i v^j h_{ij} + v^i h_{i0})dt \tag{28}$$

1. Thus the inclusion of one insertion of the $v^2 h_{00}$ operator contributes

$$\frac{1}{2}(v_1^2 + v_2^2)V_0 \tag{29}$$

2. Two insertions of the term

$$S_m = -M \int dt(1 + v^i \frac{h_{i0}(x(t))}{M_{pl}}). \tag{30}$$

which gives

$$\begin{aligned} -iV &= (-i)^2 \int dt_1 dt_2 M_1 M_2 \frac{1}{M_{pl}^2} v_1^i(t_1) v_2^j(t_2) (\langle 0 \mid T\left[h_{i0}(x_1(t_1)) h_{j0}(x_2(t_2))\right] \mid 0 \rangle. \\ &= -(-i)^2 \int dt M_1 M_2 \frac{1}{M_{pl}^2} \vec{v}_1(t) \cdot \vec{v}_2(t) \frac{1}{2} \int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_1(t) - \vec{x}_2(t))} \frac{-i}{\vec{p}^2} \\ V &= 4\vec{v}_1(t) \cdot \vec{v}_2(t) V_0 \end{aligned} \tag{31}$$

3. We can also have the first order correction to the propagator in the leading order term

$$-iV_b^{(2)} = i \int dt_1 dt_2 M_1 M_2 \frac{1}{8M_{pl}^2} \int d^4p e^{ip\cdot(x_1(\tau) - x_2(\tau))} \frac{p_0^2}{\vec{p}^4} \tag{32}$$

We can pirate our results from the QED case we worked through to find

$$\begin{aligned} V_b^{(2)} &= -\int dt_1 dt_2 M_1 M_2 \frac{[d^4p]}{8M_{pl}^2} (\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} e^{-ip_0(t_1 - t_2)}) e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)} \frac{1}{\vec{p}^4} \\ V_b^{(2)} &= -\vec{v}_1 \cdot \vec{v}_2 \frac{V_0}{2} + (\vec{v}_1 \cdot \vec{X})(\vec{v}_2 \cdot \vec{X}) \frac{V_0}{2X^2} \end{aligned} \tag{33}$$

4. We can also have one leading order insertion and one order v^2 insertion. Namely,

$$\begin{aligned} -iV &= \int d\tau_1 d\tau_2 M_1 M_2 \frac{(-i)^2}{4M_{pl}^2} \\ &\left(v_1^i(\tau_1) v_1^j(\tau_1) \langle T(h_{ij}(x_1(\tau_1) h_{00}(x_2(\tau_2))\rangle + v_2^i(\tau_2) v_2^j(\tau_2) \langle T(h_{00}(x_1(\tau_1) h_{ij}(x_2(\tau_2))\rangle\right). \\ &= (-i)^2 \int d\tau_1 d\tau_2 M_1 M_2 \frac{1}{8M_{pl}^2} (\vec{v}_1(\tau_1)^2 + \vec{v}_2(\tau_2)^2) \int [d^dp] e^{ip\cdot(x_1 - x_2)} \frac{-i}{\vec{p}^2} \\ &\approx \int dt M_1 M_2 \frac{1}{8M_{pl}^2} (\vec{v}_1(t)^2 + \vec{v}_2(t)^2) \int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_1(t) - \vec{x}_2(t))} \frac{i}{\vec{p}^2} \end{aligned} \tag{34}$$

Note no minus sign from Euclidean contraction, since it picks out the $-\eta_{\mu\nu}$ piece of the propagator. So that

$$V = (\vec{v}_1^2 + \vec{v}_2^2)V_0 \tag{35}$$

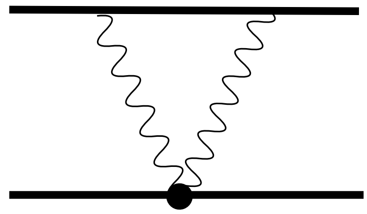


FIG. 1: Seagull contribution to the potential at order v^2.

5. Another coming from the second order term in the expansion of

$$ds = d\tau\sqrt{1 + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{h_{\mu\nu}(x(\tau))}{M_{pl}}}. \tag{36}$$

which is

$$S_m^{(2)} = i\int d\tau \frac{M}{8}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{h_{\mu\nu}(x(\tau))}{M_{pl}}\frac{dx^\rho}{d\tau}\frac{dx^\beta}{d\tau}\frac{h_{\rho\beta}(x(\tau))}{M_{pl}}. \tag{37}$$

as shown in the figure. Note the sign flip arises from $\sqrt{1+x} = 1 + x/2 - x^2/8$. Let's see how this operator scales

$$\frac{M}{M_{pl}^2}(\frac{v^{1/2}}{R})^2\frac{R}{v} \propto \frac{M}{RM_{pl}^2} \propto \frac{M}{\frac{L}{Mv}M_{pl}^2} = \frac{M^2}{M_{pl}^2}\frac{v}{L} = (Lv)\frac{v}{L} = v^2. \tag{38}$$

We consider a time ordered product of this operator with two insertions of the leading order operator

$$S_m = -iM\int d\tau(1 + \frac{1}{2}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{h_{\mu\nu}(x(\tau))}{M_{pl}}) \tag{39}$$

So that the net potential will scale as $v^2 L$. We have then (we may freely exchange t and τ at this order)

$$\begin{aligned} -iV \approx & -i\int d\tau_1 d\tau_2 d\tau_2' M_1 M_2^2 \frac{1}{8\times 2\times 2}\langle\frac{h_{00}(x_1(\tau_1))}{M_{pl}}\frac{h_{00}(x_1(\tau_1))}{M_{pl}}\frac{h_{00}(x_2(\tau_2))}{M_{pl}}\frac{h_{00}(x_2(\tau_2'))}{M_{pl}}\rangle \\ & + (1\leftrightarrow 2) \end{aligned} \tag{40}$$

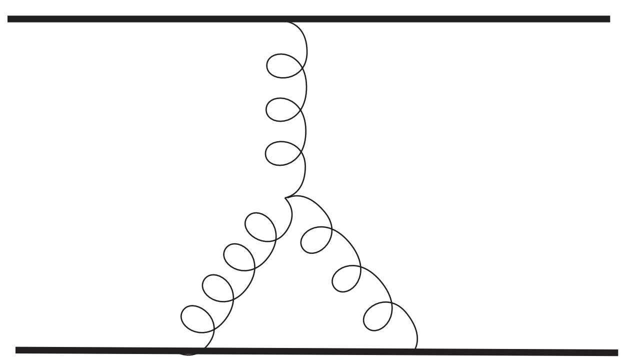
There are two possible Wick contractions which give identical contributions, but we also pick up a factor of 1/2 from each propagator

$$
\begin{aligned}
-iV \;\approx\; & \int d\tau_1 d\tau_2 d\tau_2' \frac{M_1 M_2^2}{M_{pl}^4} \frac{-i}{8\times 2\times 2\times 2\times 2}\left(\int d^4p e^{ip\cdot(x_2(\tau_2)-x_1(\tau_1))}\frac{i}{\vec{p}^2}\times\int d^4p' e^{ip'\cdot(x_2(\tau_2)-x_1(\tau_1))}\frac{i}{\vec{p'}^2}\right) \\
& + (1\leftrightarrow 2) \\
= & \int -i\frac{d\tau}{2M_1}\left(\frac{M_1M_2}{8M_{pl}^2}\int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_2(\tau)-\vec{x}_1(\tau))}\frac{i}{\vec{p}^2}\times\frac{M_1M_2}{8M_{pl}^2}\int [d^3p'] e^{-i\vec{p'}\cdot(\vec{x}_2(\tau)-\vec{x}_1(\tau))}\frac{i}{\vec{p'}^2}\right) \\
& + (1\leftrightarrow 2)
\end{aligned}
\tag{41}
$$

Let me explain the symmetry factor. There is a 1/8 from the operator, and two factors of 1/2 from the leading order operator. There is a factor of 2 from Wick contractions, but this is cancelled by a factor of 1/2 coming from the expansion of the exponential. Normally, this extra factor of 1/2 is cancelled by the permutation of the vertices, but in our case there are no external lines. Then we have two additional factors of 1/2coming from the propagators. Thus the net contribution from this sea-gull diagram is

$$
V = -\int d\tau \frac{M_1 M_2^2}{128 M_{pl}^4}\frac{1}{(4\pi)^2}\frac{1}{r^2(\tau)} \quad + (1\leftrightarrow 2). \tag{42}
$$

6. The contribution from the three graviton vertex as shown in the figure.



This calculation takes a little more work. All of the work is in determing the three graviton effective lagrangian. The idea is to split up $\sqrt{g}R$ and keep all trilinear pieces. What simplifies the calculation is the fact that the propagator with one external h_{00} graviton obeys certain simple identities

$$
P_{00:\alpha\beta}P^{00:\alpha\beta} = 1. \tag{43}
$$

$$P_{00:\alpha\beta}\eta^{\alpha\beta} = -1. \tag{44}$$

$$P_{00:\alpha\beta}P^{00:\alpha\delta} = \frac{1}{4}\delta^{\delta}_{\alpha}. \tag{45}$$

$$k^{\alpha}P_{00:\alpha\beta} = -\frac{1}{2}k_{\beta}. \tag{46}$$

We did these contractions on the computer and the net result is

$$\begin{aligned} iM \;=\; & \frac{M_1^2 M_2}{2!}(-i/(2M_{pl}))^3(-2i/M_{pl})(i)^3 \int d\tau \frac{[d^{d-1}k]}{-\vec{k}^2}\frac{[d^{d-1}p]}{-\vec{p}^2}\frac{[d^{d-1}r]}{-\vec{r}^2}\frac{-1}{8}(\vec{p}^2+\vec{k}^2+\vec{r}^2)\times \\ & e^{i\vec{k}\cdot\vec{x}_1(\tau)}e^{i\vec{p}\cdot\vec{x}_1(\tau)}e^{i\vec{r}\cdot\vec{x}_2(\tau)}\delta^3(\vec{k}+\vec{p}+\vec{r}). \end{aligned} \tag{47}$$

In reaching this point we did the r_0 energy integral. Then integrating over the other two energies forces the affine parameters to be equal when integrating over the other energy integrals. Doing the energy integrals is possible because of the NRGR potential propagator. The only integral which does not vanish is the one with $\vec{r}^2$ in the numerator. The reason is that for the other two it becomes a tadpole. That means the integral is dimensionless pure number which can be dropped. All divergent integrals are simple mass renormalizations (at this order in the PN, indeed up to 5PN for potentials, and 3PN for radiation). If one chooses to regulate the integrals by analytic continuation to d dimensions, then these integrals are simply zero. These conclusions fall out simply from the effective field theory as discussed during the work shop.

$$\begin{aligned} &= \frac{1}{8}\frac{M_1^2 M_2}{2!}(-i/(2M_{pl}))^3(-2i/M_{pl})(i)^3 \int d\tau \frac{[d^{d-1}k]}{-\vec{k}^2}\frac{[d^{d-1}p]}{-\vec{p}^2}e^{i\vec{k}\cdot(\vec{x}_1-x_2)}e^{i\vec{p}\cdot(\vec{x}_1-x_2)} \\ &= \frac{1}{8}\frac{M_1^2 M_2}{2!}(-i/(2M_{pl}))^3(-2i/M_{pl})(i)^3 \int d\tau \frac{[d^{d-1}k]}{-\vec{k}^2}\frac{[d^{d-1}p]}{-\vec{p}^2}e^{i\vec{k}\cdot(\vec{x}_1-x_2)}e^{i\vec{p}\cdot(\vec{x}_1-x_2)} \\ &= -i\frac{M_1^2 M_2}{64M_{pl}^4}V_0^2 \end{aligned} \tag{48}$$

$$-iV = -i\int d\tau \frac{M_1^2 M_2}{64M_{pl}^4}\frac{1}{(4\pi)^2}\frac{1}{r^2(\tau)} \tag{49}$$

Thus the total is

$$V = \int d\tau \frac{M_1^2 M_2}{128M_{pl}^4}\frac{1}{(4\pi)^2}\frac{1}{r^2(\tau)}. \tag{50}$$

The net sum for the velocity dependent terms (which they don't see in the static potential)

$$\begin{aligned} V^{(2)} &= -\frac{1}{2}(\vec{v}_1^2 + \vec{v}_2^2) + 4(\vec{v}_1 \cdot \vec{v}_2) - (\vec{v}_1^2 + \vec{v}_2^2) - \vec{v}_1 \cdot \vec{v}_2 \frac{1}{2} + (\vec{v}_1 \cdot \vec{X})(\vec{v}_2 \cdot \vec{X}) \frac{1}{2X^2} \\ &= -\frac{3}{2}(\vec{v}_1^2 + \vec{v}_2^2) + \frac{7}{2}(\vec{v}_1 \cdot \vec{v}_2) + (\vec{v}_1 \cdot \vec{X})(\vec{v}_2 \cdot \vec{X}) \frac{1}{2X^2} \end{aligned} \tag{51}$$

Which agrees with the result of Einstein Infeld and Hoffman. In the paper we present L instead so there is an overal minus sign.